LING 230 **Handout: Predicate Logic** **Fall 2009**
Monday, September 21st

Propositional logic doesn't go far enough with respect to the Principle of Compositionality. For example, it's clear that *is happy* expresses the same essential meaning in *Alice is happy* as it does in *Bob is happy*, but representing these sentences simply as monolithic A and B completely disguises the underlying connection between them. For this reason, we expand our formal language $\mathscr{L}$ to allow representation of units smaller than propositions.

Formal Languages with Predicates

A formal logical **language** $\mathscr{L}$ is a notational shorthand for a natural language (in this course, our object of study is English, though much of our analysis is transferable to other languages). In predicate logic, our formal language consists of individual constants, predicate letters, logical connectives, and parentheses.

Individual constants are written in $\mathscr{L}$ with lowercase letters ($a, b, c, \ldots$), and they usually translate into natural languages as names (like *Alice* and *Bob*) and definite descriptions (like *my cat*).

Predicate letters are written in $\mathscr{L}$ with uppercase letters ($A, B, C, D, \ldots$), and they usually translate into natural languages as verbs (like *admire*, a binary predicate), adjectives (like *brown*, a unary predicate), atomic sentences (like *it is cloudy*, a zero-place predicate), and nouns (like *dog*, a unary predicate).

A special binary predicate is **identity**, symbolized by $=$. By convention, this predicate is written between the individual constants, as in $a = b$ rather than in front of them. This predicate is used to represent natural language expressions denoting equality, like *is*.

Logical connectives are written in $\mathscr{L}$ with the symbols $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and $\neg$, and they usually translate into natural languages as conjunctions (like *and*, *or*, and *if…then*) and modifiers (like *not*).

Parentheses are used for grouping pieces of sentences in $\mathscr{L}$, and they can sometimes translate into natural languages as pauses or changes in intonation.

A **well-formed formula** (WFF) in $\mathscr{L}$ is defined as follows:

(1) If $a_1, \ldots, a_n$ are n individual constants and P is an n-place predicate letter, then $Pa_1 \ldots a_n$ is a WFF. (In the special case where $n = 0$, P by itself is a WFF, known as an **atomic sentence**. In the special case where P is the identity predicate $=$, $a_1 = a_2$ is a WFF.)

(2) If ϕ is any WFF, then $\neg\phi$ is also a WFF.

(3) If ϕ and ψ are any WFFs, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$ are also WFFs. (Note that since ϕ and ψ can be any WFFs, we could reverse our choices, which means that $(\psi \wedge \phi)$, $(\psi \wedge \phi)$, $(\psi \rightarrow \phi)$, and $(\psi \leftrightarrow \phi)$ are all necessarily WFFs as well.)

(4) Anything that cannot be generated by (1)–(3) in a finite number of steps is not a WFF.

By convention, the outermost parentheses of a WFF need not be written, which means that $\phi \wedge \psi$ is an acceptable shorthand for $(\phi \wedge \psi)$. Parentheses cannot be left off from the interior of a WFF; otherwise, we could not tell what the proper grouping is for something like $\phi \wedge \psi \vee \chi$.

Also by convention, $\neg(a = b)$ can be written as $a \neq b$.

The Interpretation of Formal Languages with Predicates

A **model** $\mathbb{M}$ is our formal way of defining the meaning of $\mathscr{L}$, and thus, of the natural language we are studying. A model consists of a domain of entities $\mathscr{D}$ and an interpretation function I.

Entities are identifiable, unique people and objects, whether real or imaginary, concrete or abstract. We need a way to represent entities, since we can't physically plaster them on paper. We will write an entity's usual English name in capital letters as a representation of the entity itself, e.g., NATHAN SANDERS, SANTA CLAUS, THE WHITE HOUSE, GONE WITH THE WIND, HUNGER, etc. (We could just as logically represent entities with drawings, abstract symbols, or strings taped to the paper and running out into the real world to the entity, but their English names in capital letters is a bit more practical!) The **domain** $\mathscr{D}$ of a model is a non-empty set containing all of the entities that make up the world the model describes.

The **interpretation function** I of a model assigns meaning to the individual constants and predicate letters of $\mathscr{L}$ as follows:

• I maps individual constants in $\mathscr{L}$ to entities in $\mathscr{D}$. That is, if a is an individual constant, then $I(a) \in \mathscr{D}$. For example, if b translates to *Barack Obama* (the name of the entity we know as Barack Obama), then BARACK OBAMA $\in \mathscr{D}$ and $I(b) =$ BARACK OBAMA.

• I maps atomic sentences (zero-place predicates) in $\mathscr{L}$ to truth values (either 0 or 1 in our system). That is, if S is an atomic sentence, then $I(S) \in \{0,1\}$. For example, if R translates to *it is raining*, then $I(R) = 1$ iff it really is raining, and $I(R) = 0$ otherwise (this obviously depends on the state of the world, which is precisely what the model describes).

• I maps unary (one-place) predicates in $\mathscr{L}$ to sets of individuals from $\mathscr{D}$. That is, if P is a unary predicate, then $I(P) \subseteq \mathscr{D}$. For example, if Jx translates to *x is a U.S. Supreme Court Justice*, then:

$$I(J) = \left\{ \begin{array}{l} \text{JOHN ROBERTS}, \text{CLARENCE THOMAS}, \text{SONIA SOTOMAYOR}, \\ \text{STEPHEN BREYER}, \text{ANTONIN SCALIA}, \text{ANTHONY KENNEDY}, \\ \text{RUTH BADER GINSBERG}, \text{JOHN PAUL STEVENS} \end{array} \right\}$$

(This is the interpretation in September 2009; at a different point in time, we might need a different model with a different interpretation function.)

• I maps binary predicates in $\mathscr{L}$ to sets of ordered pairs of entities from $\mathscr{D}$. That is, if P is a binary predicate, then $I(P) \subseteq \mathscr{D} \times \mathscr{D}$ (a.k.a. $\mathscr{D}^2$, the set of all ordered pairs of the form $\langle \varepsilon_1, \varepsilon_2 \rangle$, where $\varepsilon_1 \in \mathscr{D}$ and $\varepsilon_2 \in \mathscr{D}$). For example, if Gxy translates to *x is the governor of y*, then (in September 2009):

$$I(G) = \left\{ \begin{array}{l} \langle \text{ARNOLD SCHWARZENEGGER}, \text{CALIFORNIA} \rangle, \\ \langle \text{SONNY PERDUE}, \text{GEORGIA} \rangle, \\ \langle \text{DEVAL PATRICK}, \text{MASSACHUSETTS} \rangle, \\ \langle \text{PHIL BREDESEN}, \text{TENNESSEE} \rangle, \ldots \end{array} \right\}$$

In the special case of the identity predicate =, the interpretation $I(=)$ in every model is defined for every model to be the set of all pairs of entities $\langle \varepsilon, \varepsilon \rangle \subseteq \mathscr{D}^2$, i.e., all pairs of entities in which the first member and the second member are the same entity.

• In general, I maps n-ary predicates in $\mathscr{L}$ to sets of ordered n-tuples of entities from $\mathscr{D}$. That is, if P is an n-ary predicate, then $I(P) \subseteq \mathscr{D}^n$.

From $\mathbb{M}$, we can construct a valuation function $V_{\mathbb{M}}$ which assigns truth values to WFFs of $\mathscr{L}$ as follows:

• Atomic sentences are just zero-place predicates in $\mathscr{L}$, which receive truth values directly in $\mathbb{M}$ via the interpretation function I. Thus, if S is an atomic sentence, $V_{\mathbb{M}}(S) = I(S)$.

• If Pa is a unary predicate P followed by an individual constant a, then $V_{\mathbb{M}}(Pa) = 1$ iff $I(a) \in I(P)$. That is, the interpretation of a must be in the set of entities denoted by the interpretation of P.

• If Pa_1a_2 is a binary predicate P followed by two individual constants a_1 and a_2, then $V_{\mathbb{M}}(Pa_1a_2) = 1$ iff $\langle I(a_1), I(a_2)\rangle \in I(P)$. That is, the ordered pair consisting of the interpretations of a_1 and a_2 must be a member of the set of ordered pairs denoted by the interpretation of P.

In the special case of the identity predicate $=$, $V_{\mathbb{M}}(a_1 = a_2) = 1$ iff $I(a_1) = I(a_2)$.

• In general for $n > 1$, if $Pa_1 \ldots a_n$ is an n-ary predicate P followed by n individual constants $a_1, \ldots, a_n$, then $V_{\mathbb{M}}(Pa_1 \ldots a_n) = 1$ iff $\langle I(a_1), \ldots, I(a_n)\rangle \in I(P)$. That is, the ordered n-tuple consisting of the interpretations of $a_1, \ldots, a_n$ must be a member of the set of ordered n-tuples denoted by the interpretation of P.

• For WFFs built up from other WFFs using logical connectives, the truth values are the same as in propositional logic:

$$V_{\mathbb{M}}(\neg\phi) = \begin{cases} 1 & \text{iff } V_{\mathbb{M}}(\phi) = 0 \\ 0 & \text{iff } V_{\mathbb{M}}(\phi) = 1 \end{cases}$$

$$V_{\mathbb{M}}(\phi \wedge \psi) = \begin{cases} 1 & \text{iff } V_{\mathbb{M}}(\phi) = 1 \text{ and } V_{\mathbb{M}}(\psi) = 1 \\ 0 & \text{iff } V_{\mathbb{M}}(\phi) = 0 \text{ or } V_{\mathbb{M}}(\psi) = 0 \end{cases}$$

$$V_{\mathbb{M}}(\phi \vee \psi) = \begin{cases} 1 & \text{iff } V_{\mathbb{M}}(\phi) = 1 \text{ or } V_{\mathbb{M}}(\psi) = 1 \\ 0 & \text{iff } V_{\mathbb{M}}(\phi) = 0 \text{ and } V_{\mathbb{M}}(\psi) = 0 \end{cases}$$

$$V_{\mathbb{M}}(\phi \rightarrow \psi) = \begin{cases} 1 & \text{iff } V_{\mathbb{M}}(\phi) = 0 \text{ or } V_{\mathbb{M}}(\psi) = 1 \\ 0 & \text{iff } V_{\mathbb{M}}(\phi) = 1 \text{ and } V_{\mathbb{M}}(\psi) = 0 \end{cases}$$

$$V_{\mathbb{M}}(\phi \leftrightarrow \psi) = \begin{cases} 1 & \text{iff } V_{\mathbb{M}}(\phi) = V_{\mathbb{M}}(\psi) \\ 0 & \text{iff } V_{\mathbb{M}}(\phi) \neq V_{\mathbb{M}}(\psi) \end{cases}$$

Quick Notes on Sets

A **set** is an unordered collection of **elements** (which could be anything: numbers, names, other sets, etc.). The list of elements of a set is enclosed in curly braces. For example, if S if the set of possible values on a six-sided die, then $S = \{1,2,3,4,5,6\}$. Since sets are unordered, the sets $\{a,b,c\}$ and $\{c,b,a\}$ are equal. Note: because sets are unordered collections, the sets $\{a,a,a\}$ and $\{a\}$ are equivalent, with $\{a\}$ being the preferred notation. If a is an element in the set A, we write $a \in A$. If a is not an element in the set A, we write $a \notin A$. The special set containing no elements is called the **empty set** or the **null set**, which we symbolize with $\varnothing$ or $\{\ \}$.

If every element of set A is also an element in set B, we say that A is a **subset** of B, and we write $A \subseteq B$. If A is not a subset of B, we write $A \not\subseteq B$. Note that by definition, every set is a subset of itself. Also by definition, the empty set is a subset of every set.

An n**-tuple** is a special type of collection in which the elements are ordered and repetition is meaningful. An **ordered pair** is an n-tuple with two ordered elements, an **ordered triplet** is an n-tuple with three ordered elements, etc. n-tuples are written with angle brackets $\langle\ \rangle$ instead of curly braces. For example, $\langle 2,2,5\rangle$ is the ordered triplet with 2 as the first element, 2 as the second, and 5 as the third. For any set S, S^n is the set of all ordered n-tuples that can be built from the elements of S. If $S = \{1,2,3\}$, then $S^2 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$.

The **union** of two sets A and B is the set containing every member of A and every member of B, written $A \cup B$. $A \cup \emptyset = A$ since the empty set contains no new members to add to the union. The **intersection** of A and B is the set containing every member that is common to both A and B, written $A \cap B$. If A and B have no members in common, then $A \cap B = \emptyset$.