LING 230 **Handout: Predicate Logic with Quantification** **Fall 2009**

Monday, September 28th

Formal Languages with Quantifiers

In order to add the power of quantification to our ability to compute the meaning of sentences, our formal language $\mathscr{L}$ needs to have two new connectives: a **universal quantifier** $\forall$ (which translates to *every*, *each*, *all*, etc. in English) and an **existential quantifier** $\exists$ (which translates to *a*, *some*, *(at least) one*, etc. in English). In addition, $\mathscr{L}$ needs to have **variables** like x, y, and z (which can sometimes translate to pronouns in English). With new components available to build to sentences, a new definition of a WFF is required. We begin by defining **formula** in $\mathscr{L}$:

(1) If $\tau_1, \ldots, \tau_n$ are n **terms** (individual constants and variables) and P is an n-place predicate letter, then $P\tau_1 \ldots \tau_n$ is a formula. (In the special case where $n = 0$, P by itself is a formula, known as an **atomic formula**. In the special case where P is the identity predicate $=$, by convention, $\tau_1 = \tau_2$ is a formula.)

(2) If ϕ is any formula, then $\neg\phi$ is also a formula.

(3) If ϕ and ψ are any formulas, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$ are also formulas. (Note that since ϕ and ψ can be any formulas, we could reverse our choices, which means that $(\psi \wedge \phi)$, $(\psi \wedge \phi)$, $(\psi \rightarrow \phi)$, and $(\psi \leftrightarrow \phi)$ are all necessarily formulas as well.)

(4) If ϕ is any formula and x is a variable, then $\forall x\phi$ and $\exists x\psi$ are both formulas.

(5) Anything that cannot be generated by (1)–(4) in a finite number of steps is not a formula.

For any quantified formula of the form $\forall x\phi$ or $\exists x\phi$, ϕ is called the **scope** of the quantifier. If a variable x is not within the scope of a quantifier $\forall x$ or $\exists x$ (with a matching variable marker), then the variable is said to be **free**. If the variable x is free in the formula ϕ, then in a quantified formula like $\forall x\phi$ or $\exists x\phi$, x is said to be **bound** by the quantifier $\forall x$ or $\exists x$. A **well-formed formula** (WFF) (or **sentence**) in $\mathscr{L}$ is any formula which has no free variables; that is, all variables in a WFF must be bound. For example:

$$\begin{array}{ccc} \text{not a WFF} & \text{not a WFF} & \text{WFF} \\[1em] \forall x\,\overbrace{(Px \wedge T\underset{\substack{\uparrow\\ \text{free}}}{y})}^{\text{scope of } \forall x} \wedge \neg F\underset{\substack{\uparrow\\ \text{free}}}{x} & \exists y\,\overbrace{(P\underset{\substack{\uparrow\\ \text{free}}}{x} \wedge Ty)}^{\text{scope of } \exists y} \wedge \neg F\underset{\substack{\uparrow\\ \text{free}}}{x} & \forall x\,\overbrace{\left(\exists y\,\underbrace{(Px \wedge Ty)}_{\text{scope of } \exists y} \wedge \neg Fx\right)}^{\text{scope of } \forall x} \end{array}$$

The Interpretation of Formal Languages with Predicates

To calculate the valuation of a quantified sentence in a model $\mathbb{M}$, we need to extend the definition of $V_{\mathbb{M}}$ to give values for the sentences $\forall x\phi$ and $\exists x\phi$.

$$V_{\mathbb{M}}(\forall x\phi) = \begin{cases} 1 & \text{iff } V_{\mathbb{M}}([\alpha/x]\phi) = 1 \text{ for all individual constants } \alpha \text{ in } \mathscr{L} \\ 0 & \text{iff } V_{\mathbb{M}}([\alpha/x]\phi) = 0 \text{ for at least one individual constant } \alpha \text{ in } \mathscr{L} \end{cases}$$

$$V_{\mathbb{M}}(\exists x\phi) = \begin{cases} 1 & \text{iff } V_{\mathbb{M}}([\alpha/x]\phi) = 1 \text{ for at least one individual constant } \alpha \text{ in } \mathscr{L} \\ 0 & \text{iff } V_{\mathbb{M}}([\alpha/x]\phi) = 0 \text{ for all individual constants } \alpha \text{ in } \mathscr{L} \end{cases}$$

We use the notation $[\alpha/x]\phi$ to indicate replacement of every instance of the variable x in the formula with the individual constant α. For example, if ϕ is the formula $Py \wedge Fby$, then $[c/y]\phi$ would be the sentence $Pc \wedge Fbc$, with all occurrences of y replaced by c.

This extension to the definition of $V_{\mathbb{M}}$ assumes that every entity in $\mathscr{D}$ has a name in $\mathscr{L}$ assigned to it withn $\mathbb{M}$ by the interpretation function. That is, for every entity ε, there is an individual constant α in $\mathscr{L}$ such that $I(\alpha) = \varepsilon$. A stronger definition for the valuation of quantified sentences can be created to account for models in which $\mathscr{D}$ contains entities that do not have a name in $\mathscr{L}$ assigned to them by I (this stronger definition basically allows the nameless entities to be temporarily assigned names). For simplicity, we will not use any models with nameless entities, so we won't need the stronger definition.

Examples with Quantification

$$\mathscr{L}: \quad \begin{array}{rl rl rl rl} g: & \textit{Garfield} & r: & \textit{Ren} & Ax: & \textit{x is an animal} & Mx: & \textit{x is a mouse} \\ j: & \textit{Jerry} & p: & \textit{Pluto} & Cx: & \textit{x is a cat} & Tx: & \textit{x can talk} \\ m: & \textit{Mickey} & s: & \textit{Scratchy} & Dx: & \textit{x is a dog} & Fxy: & \textit{x is is friends with y} \end{array}$$

$$\mathbb{M}: \quad \mathscr{D} = \left\{ \begin{array}{c} \textsc{Garfield}, \textsc{Jerry}, \textsc{Mickey}, \\ \textsc{Ren}, \textsc{Pluto}, \textsc{Scratchy} \end{array} \right\} \qquad \begin{array}{rcl} I(g) & = & \textsc{Garfield} \\ I(j) & = & \textsc{Jerry} \\ I(m) & = & \textsc{Mickey} \end{array} \qquad \begin{array}{rcl} I(r) & = & \textsc{Ren} \\ I(p) & = & \textsc{Pluto} \\ I(s) & = & \textsc{Scratchy} \end{array}$$

$$I(A) = \mathscr{D} \qquad I(C) = \left\{ \begin{array}{c} \textsc{Garfield}, \\ \textsc{Scratchy} \end{array} \right\} \qquad I(D) = \left\{ \begin{array}{c} \textsc{Ren}, \\ \textsc{Pluto} \end{array} \right\}$$

$$I(M) = \left\{ \begin{array}{c} \textsc{Jerry}, \\ \textsc{Mickey} \end{array} \right\} \qquad I(T) = \left\{ \begin{array}{c} \textsc{Mickey}, \\ \textsc{Ren}, \\ \textsc{Scratchy} \end{array} \right\}$$

$$I(F) = \left\{ \begin{array}{c} \langle \textsc{Ren}, \textsc{Scratchy} \rangle, \langle \textsc{Scratchy}, \textsc{Ren} \rangle, \langle \textsc{Ren}, \textsc{Jerry} \rangle, \langle \textsc{Jerry}, \textsc{Ren} \rangle, \\ \langle \textsc{Pluto}, \textsc{Mickey} \rangle, \langle \textsc{Mickey}, \textsc{Pluto} \rangle, \langle \textsc{Jerry}, \textsc{Mickey} \rangle, \langle \textsc{Mickey}, \textsc{Jerry} \rangle \end{array} \right\}$$

(1) Dogs and cats are animals.

(2) Dogs are not cats.

(3) Ren is friends with a mouse.

(4) Ren is friends with all mice.

(5) Ren isn't friends with any mice.

(6) A dog is friends with Scratchy.

(7) No dog is friends with Scratchy.

(8) Some dog is friends with a cat.

(9) No dog is friends with a cat.

(10) Some dog is friends with no cats.

(11) Every dog is friends with a cat.

(12) A talking dog is friends with a cat.

(13) No talking dog is friends with a cat.

(14) Some talking dog is friends with a talking cat.

(15) No talking dog is friends with a talking cat.

(16) Some talking dog is friends with no talking cats.

(17) Every talking dog is friends with a talking cat.

(18) Ren is friends with a cat's friend.

(19) Ren is friends with a cat's talking friend.

(20) Ren is friends with someone else's friend.

(21) Ren is no one's friend.

(22) Every cat is friends with someone.

Calculation of $V_{\mathbb{M}}(22)$:

$V_{\mathbb{M}}\big(\forall x(Cx \rightarrow \exists yFxy)\big) = 1$ iff $V_{\mathbb{M}}(C\alpha \rightarrow \exists yF\alpha y) = 1$, for all individual constants α in $\mathscr{L}$. That is, in order to prove that (22) is true, we would need to show that $C\alpha \rightarrow \exists yF\alpha y$ is true for every possible replacement for α, which is all of the individual constants g, j, m, r, p, and s.

Conversely, we could prove instead that (22) is false by finding at least one individual constant to substitute for α that will make $C\alpha \rightarrow \exists yF\alpha y$ be false. By inspection of the model, we can tell that there is exactly one individual constant that makes this sentence false, g, because Garfield is the only cat that isn't friends with anyone. So, we need to let $\alpha = g$, and show that $Cg \rightarrow \exists yFgy$ is false to show that (22) is also false. $V_{\mathbb{M}}(Cg \rightarrow \exists yFgy) = 0$ iff $V_{\mathbb{M}}(Cg) = 1$ and $V_{\mathbb{M}}(\exists yFgy) = 0$. Let's show each of these in order.

First, $V_{\mathbb{M}}(Cg) = 1$ iff $I(g) \in I(C)$, which is the case iff $\text{G}\textsc{arfield} \in I(C)$. This is indeed the case, so $V_{\mathbb{M}}(Cg) = 1$. Call this result $\boxed{\text{A}}$.

Next, $V_{\mathbb{M}}(\exists yFgy) = 0$ iff $V_{\mathbb{M}}(Eg\beta) = 0$, for all individual constants β in $\mathscr{L}$. That is, we need to show that $Fg\beta$ is false for every possible replacement for β, which is all of the individual constants g, j, m, r, p, and s. (β is used here to avoid confusion with the previous α.)

We start with $\beta = j$. $V_{\mathbb{M}}(Fgj) = 0$ iff $\langle I(g), I(j)\rangle \notin I(F)$, which is the case iff $\langle \text{G}\textsc{arfield}, \text{J}\textsc{erry}\rangle \notin I(F)$. This is indeed the case, so $V_{\mathbb{M}}(Fgj) = 0$.

Next is $\beta = m$. $V_{\mathbb{M}}(Fgm) = 0$ iff $\langle I(g), I(m)\rangle \notin I(F)$, which is the case iff $\langle \text{G}\textsc{arfield}, \text{M}\textsc{ickey}\rangle \notin I(F)$. This is indeed the case, so $V_{\mathbb{M}}(Fgm) = 0$.

(This must be repeated for each of $\beta = r$, $\beta = p$, $\beta = s$, and even $\beta = g$! In each case, we would show that $V_{\mathbb{M}}(Fg\beta) = 0$.)

$\vdots$

Since $V_{\mathbb{M}}(Fg\beta) = 0$, for all individual constants β in $\mathscr{L}$, then $V_{\mathbb{M}}(\exists yFgy) = 0$. Call this result $\boxed{\text{B}}$.

From $\boxed{\text{A}}$ we have $V_{\mathbb{M}}(Cg) = 1$, and from $\boxed{\text{B}}$ we have $V_{\mathbb{M}}(\exists yFgy) = 0$, so $V_{\mathbb{M}}(Cg \rightarrow \exists yFgy) = 0$, which means that $V_{\mathbb{M}}(C\alpha \rightarrow \exists yF\alpha y) = 0$ for at least one individual constant α in $\mathscr{L}$ (namely, $\alpha = g$). Thus, $V_{\mathbb{M}}\big(\forall x(Cx \rightarrow \exists yFxy)\big) = 0$.

Final Reminders about Quantification

Existential quantification usually translates into $\mathscr{L}$ with an obligatory conjunction $\wedge$, while universal quantification usually translates into $\mathscr{L}$ with an obligatory material implication $\rightarrow$, with the quantified predicates (the **restriction**) appearing in the antecedent of the material implication.

Proving an existentially quantified sentence true is easy. All you have to do is find one single individual constant that makes the sentence true in the model. You are very likely going to be asked to do this many times. However, proving an existentially quantified sentence false is hard. You have to prove that it is false for every single individual constant in the language. Unless the model is small, you will usually not be asked to give calculations for proving the falsity of an existentially quantified sentence.

In contrast, proving a universally quantified sentence false is easy. All you have to do is find one single individual constant that makes the sentence false in the model. You are very likely going to be asked to do this many times. However, proving a universally quantified sentence true is hard. You have to prove that it is true for every single individual constant in the language. Again, unless the model is small, you will usually not be asked to give calculations for proving the truth of a universally quantified sentence.

Two Sample Proofs

For both of these theorems, assume that P and Q are unary predicates in $\mathscr{L}$, and that $\mathbb{M}$ is a model of $\mathscr{L}$ with a domain $\mathscr{D}$ and an interpretation function I.

Theorem: If $I(P) \cap I(Q) = \varnothing$ then $V_{\mathbb{M}}\big(\exists x(Px \wedge Qx)\big) = 0$.

Proof: Let $I(P) \cap I(Q) = \varnothing$. The intersection $I(P) \cap I(Q)$ is the set that contains all entities that are each members of both $I(P)$ and $I(Q)$. Since the empty set has no members, there is no entity that is a member of the intersection of $I(P)$ and $I(Q)$, which means that there is no entity that is a member of both $I(P)$ and $I(Q)$.

$V_{\mathbb{M}}\big(\exists x(Px \wedge Qx)\big) = 0$ iff $V_{\mathbb{M}}(P\alpha \wedge Q\alpha) = 0$ for all individual constants α, including a completely arbitrary constant $\overline{\alpha}$. $V_{\mathbb{M}}(P\overline{\alpha} \wedge Q\overline{\alpha}) = 0$ iff $V_{\mathbb{M}}(P\overline{\alpha}) = 0$ or $V_{\mathbb{M}}(Q\overline{\alpha}) = 0$, which happens iff $I(\overline{\alpha}) \notin I(P)$ or $I(\overline{\alpha}) \notin I(Q)$. Let $I(\overline{\alpha}) = \overline{\varepsilon}$ be whatever entity $\overline{\alpha}$ happens to refer to in $\mathbb{M}$ (we assume that all individual constants do in fact have a valid interpretation in every model). Then $V_{\mathbb{M}}\big(\exists x(Px \wedge Qx)\big) = 0$ iff $\overline{\varepsilon} \notin I(P)$ or $\overline{\varepsilon} \notin I(Q)$. $\boxed{\text{A}}$

For any given entity like $\overline{\varepsilon}$ and any given set like $I(P)$, we have two cases to consider: (1) $\overline{\varepsilon} \in I(P)$ and (2) $\overline{\varepsilon} \notin I(P)$.

Consider case (1), in which $\overline{\varepsilon} \in I(P)$. Since no entity can be a member of both $I(P)$ and $I(Q)$, we know that $\overline{\varepsilon} \notin I(Q)$. This satisfies the second condition in $\boxed{\text{A}}$, so we know that $V_{\mathbb{M}}(P\overline{\alpha} \wedge Q\overline{\alpha}) = 0$ in case (1).

Consider case (2), in which $\overline{\varepsilon} \notin I(P)$. This satisfies the first condition in $\boxed{\text{A}}$, so we also know that $V_{\mathbb{M}}(P\overline{\alpha} \wedge Q\overline{\alpha}) = 0$ in case (2).

Thus, in both possible cases stemming from our assumption, it is necessary for $V_{\mathbb{M}}(P\overline{\alpha} \wedge Q\overline{\alpha}) = 0$. Since our choice of $\overline{\alpha}$ was arbitrary (nothing in this proof depends on special properties of $\overline{\alpha}$), this means that $V_{\mathbb{M}}(P\alpha \wedge Q\alpha) = 0$ for all choices of α, which means that $V_{\mathbb{M}}\big(\exists x(Px \wedge Qx)\big) = 0$.

Theorem: If $I(P) \cup I(Q) = \mathscr{D}$ then $V_{\mathbb{M}}\big(\forall x(Px \vee Qx)\big) = 1$.

Proof: Let $I(P) \cup I(Q) = \mathscr{D}$. The union $I(P) \cup I(Q)$ is the set that contains all entities that are each members of either $I(P)$ or $I(Q)$. Since $\mathscr{D}$ contains all entities, every entity is a member of the union of $I(P)$ and $I(Q)$, which means that every entity is a member of either $I(P)$ or $I(Q)$.

$V_{\mathbb{M}}\big(\forall x(Px \vee Qx)\big) = 1$ iff $V_{\mathbb{M}}(P\alpha \vee Q\alpha) = 1$ for all individual constants α, including a completely arbitrary constant $\overline{\alpha}$. $V_{\mathbb{M}}(P\overline{\alpha} \vee Q\overline{\alpha}) = 1$ iff $V_{\mathbb{M}}(P\overline{\alpha}) = 1$ or $V_{\mathbb{M}}(Q\overline{\alpha}) = 1$, which happens iff $I(\overline{\alpha}) \in I(P)$ or $I(\overline{\alpha}) \in I(Q)$. Let $I(\overline{\alpha}) = \overline{\varepsilon}$ be whatever entity $\overline{\alpha}$ happens to refer to in $\mathbb{M}$. Then $V_{\mathbb{M}}\big(\forall x(Px \vee Qx)\big) = 1$ iff $\overline{\varepsilon} \in I(P)$ or $\overline{\varepsilon} \in I(Q)$. $\boxed{\text{A}}$

We have two cases to consider: (1) $\overline{\varepsilon} \in I(P)$ and (2) $\overline{\varepsilon} \notin I(P)$.

Consider case (1), in which $\overline{\varepsilon} \in I(P)$. This satisfies the first condition in $\boxed{\text{A}}$, so we know that $V_{\mathbb{M}}(P\overline{\alpha} \vee Q\overline{\alpha}) = 1$ in case (1).

Consider case (2), in which $\overline{\varepsilon} \notin I(P)$. Since every entity must be a member of either $I(P)$ or $I(Q)$, we know that $\overline{\varepsilon} \in I(Q)$. This satisfies the second condition in $\boxed{\text{A}}$, so we know that $V_{\mathbb{M}}(P\overline{\alpha} \vee Q\overline{\alpha}) = 1$ in case (2).

Thus, in both possible cases stemming from our assumption, it is necessary for $V_{\mathbb{M}}(P\overline{\alpha} \vee Q\overline{\alpha}) = 1$. Since our choice of $\overline{\alpha}$ was arbitrary, this means that $V_{\mathbb{M}}(P\alpha \vee Q\alpha) = 1$ for all choices of α, which means that $V_{\mathbb{M}}\big(\forall x(Px \vee Qx)\big) = 1$.