

Thursday, October 8th

Recall that the *ex falso sequitur quodlibet* (a.k.a. EFSQ) rule allows us to derive any sentence we want if we already have reached a contradiction. That is, $\perp \vdash \phi$. As stated in class, the validity of this rule is conceptually related to why $P \rightarrow Q$ is true when P is false, and why $\forall x(Px \rightarrow Qx)$ is true when $I(P)$ is empty. If you had trouble seeing the validity of EFSQ in lecture, hopefully one of the following short discussions will help give you a better understanding of the validity of EFSQ.

Lack of a counterexample. I claim that EFSQ is a valid argument. By definition, this means that in every model \mathbb{M} in which $V_{\mathbb{M}}(\perp) = 1$, we are guaranteed that $V_{\mathbb{M}}(\phi) = 1$, for every sentence ϕ .

Like any good skeptical student, you disagree with me, because you think EFSQ is invalid. This means that you think that there is some particular model \mathbb{M}^* and some particular sentence ϕ^* that I forgot to look at that disproves my claim.

I ask you to tell me what this particular \mathbb{M}^* looks like. Obviously, it must have $V_{\mathbb{M}^*}(\perp) = 1$ and $V_{\mathbb{M}^*}(\phi^*) = 0$ to falsify my claim. But this is impossible, because $\mathbb{M}^*(\perp) = 1$ violates the definition of contradictions (which must be false in every model, including \mathbb{M}^*). Thus, your hypothetical counterexample \mathbb{M}^* cannot exist. Since you have failed to come up with a counterexample, my original claim is true, so EFSQ is indeed valid!

A simple argument in natural deduction. The sentence $A \wedge \neg A$ is a known contradiction. From this specific contradiction, we can derive any sentence ϕ that we want to, using three ordinary, intuitive rules of natural deduction (elimination of \wedge , introduction of \vee , and disjunctive syllogism):

1	$A \wedge \neg A$	assumption
2	A	$E\wedge, 1$
3	$A \vee \phi$	$I\vee, 2$
4	$\neg A$	$E\wedge, 1$
5	ϕ	$DS, 3, 4$

Of course, this argument works because of the particular contradiction we used an assumption. But since all contradictions are logically equivalent (since they are all necessarily false), it doesn't matter which one we start with!

Mathematical example. The mathematical equation $1 = 0$ is a known contradiction. From this contradiction, we can derive any arbitrary mathematical equation $x = y$, using ordinary rules of algebra, including the fact that equations are transitive (i.e., if $x = z$ and $y = z$, then $x = y$):

$1 = 0$	starting equation
$x = 0$	multiply both sides of $1 = 0$ by x
$y = 0$	multiply both sides of $1 = 0$ by y
$x = y$	transitivity (from $x = 0$ and $y = 0$)

Another mathematical example. Suppose we want to use $1 = 0$ to prove something other than just an equation, an arbitrary general claim C , which could be anything like “ $\sqrt{2}$ is irrational”, “all squares are round”, or “some unicorns have two horns”. Define the function $f(C)$ as follows:

$$f(C) = \begin{cases} 1 & \text{iff claim } C \text{ is true} \\ 0 & \text{otherwise} \end{cases}$$

Case (1): If $f(C) = 1$, then C is true. Case (2): If $f(C) = 0$, then because $1 = 0$ and equations are transitive, we know that $f(C) = 1$, which means that C is true. Thus, for both possible values of $f(C)$, C turns out to be true, so C must indeed be true. . . regardless of what C actually claims. (We can of course show that C is also false!)