LING 230 **Handout: Rules Requiring Subarguments (from Gamut)** **Fall 2009**

Monday, October 19th

These rules rely on using all the previous lines in the argument, $\chi_i,\ldots,\chi_j$, plus one new assumption ϕ, entered into the argument within a subargument. While $\chi_i,\ldots,\chi_j$ are part of the main argument, ϕ is not. It is a hypothetical conjecture, and the subargument follows the logical consequences of seeing what can be derived by temporarily pretending that ϕ could have been part of the main argument.

$\boxed{\boxed{\mathrm{I}\!\to}}$ introduction rule for →,

a.k.a. $\boxed{\boxed{\mathrm{CP}}}$ conditional proof

if $\chi_i,\ldots,\chi_j,\phi \vdash \psi$,

then $\chi_i,\ldots,\chi_j \vdash \phi \to \psi$ (p.132)

i	χ_i	
	$\vdots$	
j	χ_j	
$j+1$	ϕ	assumption
	$\vdots$	
k	ψ	
$k+1$	$\phi \to \psi$	I→ (or CP)

$\boxed{\boxed{\mathrm{I}\neg}}$ introduction rule for ¬,

a.k.a. $\boxed{\boxed{\mathrm{RAA}}}$ *reductio ad absurdum*

if $\chi_i,\ldots,\chi_j,\phi \vdash \bot$,

then $\chi_i,\ldots,\chi_j \vdash \neg\phi$ (p.138)

i	χ_i	
	$\vdots$	
j	χ_j	
$j+1$	ϕ	assumption
	$\vdots$	
k	$\bot$	
$k+1$	$\neg\phi$	I¬ (or RAA)

Any line from any argument (main or sub) may be used to justify a line in a lower subargument, no matter how deeply embedded, as long as the argument the line comes from has not yet reached its conclusion. However, once a subargument has reached its conclusion, all of its lines are completely inaccessible and cannot be used to justify a later line. Otherwise, we could simply assume anything we wanted, and then repeat it later in the main argument!

It's possible to use these rules without any previous lines at all. In this way, we can derive tautologies with no assumptions in the main argument. For example, we can show $\vdash \phi \to \phi$ and $\vdash \neg(\phi \wedge \neg\phi)$:

1	ϕ	assumption
2	ϕ	Rep, 1
3	$\phi \to \phi$	I→/CP

1	$\phi \wedge \neg\phi$	assumption
2	$\neg\phi$	E∧, 1
3	ϕ	E∧, 1
4	$\bot$	E¬, 2, 3
5	$\neg(\phi \wedge \neg\phi)$	I¬/RAA

I→/CP is often used to create the implications needed for rules like E∨ and CD. While I→/CP can in principle be used to set up the usage of MP, note that it would not actually be necessary, since in order for MP to work, you would need to be able to derive ϕ as a line in the main argument already, so there would be no point in assuming it as the premise of a subargument.

I¬/RAA (also known as "proof by contradiction") is one of the most useful, all-purpose logical rules. Whenever you find yourself stuck in a logical argument, you can always use RAA by assuming the negation of what you are actually trying to prove. The resulting argument may end up being longer and more convoluted than what might have otherwise been necessary, but it may often be more obvious how to proceed when you are using RAA, especially when armed with equivalence laws.

Equivalence Laws (not in Gamut)

Two sentences ϕ and ψ are **logically equivalent** if $V_{\mathbb{M}}(\phi) = V_{\mathbb{M}}(\psi)$ for all models $\mathbb{M}$. We can write $\phi \equiv \psi$ for short.

If $\phi \equiv \psi$, it is clear that $\phi \models \psi$ and $\psi \models \phi$, since in any given model, ϕ and ψ are either both true or both false. Because our system of natural deduction is complete (a result we have not proven but are simply accepting as true), this means that $\phi \vdash \psi$ and $\psi \vdash \phi$, so we can always derive one from the other in a logical argument. The following equivalence laws are common enough that they have names:

DN	double negation equivalence law:	$\neg\neg\phi \equiv \phi$
Comm	commutative equivalence law:	$\phi \wedge \psi \equiv \psi \wedge \phi$ $\phi \vee \psi \equiv \psi \vee \phi$ $\phi \leftrightarrow \psi \equiv \psi \leftrightarrow \phi$
Assoc	associative equivalence law:	$(\phi \wedge \psi) \wedge \chi \equiv \phi \wedge (\psi \wedge \chi)$ $(\phi \vee \psi) \vee \chi \equiv \phi \vee (\psi \vee \chi)$
Dist	distributive equivalence law:	$\phi \wedge (\psi \vee \chi) \equiv (\phi \wedge \psi) \vee (\phi \wedge \chi)$ $\phi \vee (\psi \wedge \chi) \equiv (\phi \vee \psi) \wedge (\phi \vee \chi)$
DeM	DeMorgan's equivalence law:	$\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$ $\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$
Contra	contrapositive equivalence law:	$\phi \rightarrow \psi \equiv \neg\psi \rightarrow \neg\phi$
Exp	exportation equivalence law:	$(\phi \wedge \psi) \rightarrow \chi \equiv \phi \rightarrow (\psi \rightarrow \chi)$ $(\phi \wedge \psi) \rightarrow \chi \equiv \psi \rightarrow (\phi \rightarrow \chi)$
Impl	implication equivalence law:	$\phi \rightarrow \psi \equiv \neg\phi \vee \psi$
Equiv	equivalence equivalence law:	$\phi \leftrightarrow \psi \equiv (\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi)$

It can be shown that if $\phi \equiv \psi$, and ϕ is a subformula of χ, then $\chi \equiv [\psi/\phi]\chi$. That is, χ's truth value will not change if ϕ is replaced by ψ. Given the completeness of natural deduction, these equivalence laws can therefore be used to create new lines in an argument not only by operating on an entire previous line, but also by replacing just a part of a previous line. For example, if $\exists x\big(Px \rightarrow \neg(Qx \wedge Rx)\big)$ is a line in an argument, then DeM allows us to derive $\exists x\big(Px \rightarrow (\neg Qx \vee \neg Rx)\big)$ on a later line. The proof that we can make equivalence replacements within a line is a bit complex, especially since we need to take into account the possibility of replacing a non-WFF that contains variables.