LING 230 **Handout: Quantifier Equivalence Law and Rules** **Fall 2009**

Monday, November 2nd

Recall that two sentences ϕ and ψ are defined as logically equivalent ($\phi \equiv \psi$) when $V_{\mathbb{M}}(\phi) = V_{\mathbb{M}}(\psi)$ for all models $\mathbb{M}$. The following equivalence law shows the connection between the two quantifiers and negation:

QN quantifier negation equivalence law (cf. p.100): $\neg\exists v\phi \equiv \forall v\neg\phi$
$\neg\forall v\phi \equiv \exists v\neg\phi$
where v is any variable

As with other equivalence laws, QN can be used to derive a new line in argument, even by replacing only a part of a previous line, because if $\phi \equiv \psi$, and ϕ is a subformula of χ, then $\chi \equiv [\psi/\phi]\chi$.

There are also four rules of natural deduction allowing for the introduction and elimination of $\forall$ and $\exists$. The first two rules listed below follow straightforwardly from the obvious meanings of the quantifiers and the connection between variables and individual constants:

I∃ introduction rule for $\exists$,
a.k.a. **EG** existential generalization
$[\alpha/v]\phi \vdash \exists v\phi$, where
α is any individual constant, and
where v is any variable (p.142)

i	$[\alpha/v]\phi$	
	$\vdots$	
j	$\exists v\phi$	I∃/EG, i

E∀ introduction rule for $\forall$,
a.k.a. **UI** universal instantiation
$\forall v\phi \vdash [\alpha/v]\phi$, where
α is any individual constant, and
where v is any variable (p.142–143)

i	$\forall v\phi$	
	$\vdots$	
j	$[\alpha/v]\phi$	E∀/UI, i

I∃/EG allows us to introduce $\exists$ by existentially generalizing over a given individual constant. This is because a sentence that is true for a specific entity is also true for *some* entity. For example, from Hp, we can derive $\exists xHx$, as well as $\exists yHy$, $\exists zHz$, etc. (for example, if Pat is happy, then someone must be happy).

Any number of occurrences of the same constant can be existentially generalized with one use of I∃/EG because the existential quantifier is being introduced, so leaving occurrences of the individual constant will still result in a WFF. For example, $Pa \wedge Qa$ can be generalized as $\exists x(Px \wedge Qa)$, $\exists x(Pa \wedge Qx)$, and $\exists x(Px \wedge Qx)$. We can also generalize over different occurrences of the same constant with different uses of I∃/EG. For example, $Pa \wedge Qa$ can be generalized as $\exists x(Px \wedge Qa)$, which can then be generalized $\exists y\exists x(Px \wedge Qy)$.

E∀/UI allows us to eliminate $\forall$ by instantiating a universally quantified variable as any individual constant of our choice. This is because a universally quantified sentence is true for all entities, so it is true for *any* entity we choose. For example, from $\forall xHx$, we can derive Hp, as well as $H\ell$, Hc, Hf, etc. (for example, if everyone is happy, then Pat must be happy, as must Lee, Chris, Frankie, etc.).

Every occurrence of the universally quantified variable must be instantiated with each use of E∀/UI, because the universal quantifier is being eliminated, so any remaining occurrences of the variable would be free which would not result in a WFF. For example, $\forall xPx \wedge Qx$ must be instantiated as $Pa \wedge Qa$ (or $Pb \wedge Qb$, etc.), not as $Pa \wedge Qx$ or $Px \wedge Qa$.

The next two rules require an **arbitrary** individual constant, which means that the constant cannot appear in certain positions in previous lines of the argument.

I∀ introduction rule for ∀, a.k.a. **UG** universal generalization

$[\alpha/\nu]\phi \vdash \forall\nu\phi$, α is an arbitrary individual constant that does not appear in ϕ or in any accessible assumption for line j, and where ν is any variable (p.143)

i	$[\alpha/\nu]\phi$	
	$\vdots$	
j	$\forall\nu\phi$	I∀/UG, i

I∀/UG allows us to introduce ∀ by universally generalizing over an arbitrarily chosen individual constant. This is because a sentence that is true for an arbitrary entity is also true for *all* entities. For example, if we can derive Hp just as easily as $H\ell$, Hc, Hf, etc., for any individual constant, then we can derive $\forall xHx$ (for example, if an arbitrarily chosen person is happy, then everyone must be happy).

Every occurrence of the arbitrary individual constant must be universally generalized at the same time. For example, $Pa \wedge Qa$ must be generalized as $\forall x(Px \wedge Qx)$, not as $\forall x(Pa \wedge Qx)$ or $\forall x(Px \wedge Qa)$.

I∀/UG is typically used after having used rules like E∀/UI, I¬/RAA, I→/CP, or EFSQ, since these rules allow for the creation of new sentences containing an arbitrary constant.

E∃ elimination rule for ∃, a.k.a. **EI** existential instantiation

$\exists\nu\phi, [\alpha/\nu]\phi \to \psi \vdash \psi$, where α is an arbitrary individual constant that does not appear in ϕ, in ψ, or in any accessible assumption for line k, and where ν is any variable (p.145)

i	$\exists\nu\phi$	
	$\vdots$	
j	$[\alpha/\nu]\phi \to \psi$	
	$\vdots$	
k	ψ	E∃/EI, i, j

E∃/EI is similar in concept to E∨, allowing us to eliminate ∃ indirectly through use of an implication. This is because an existentially quantified sentence is true for some entity, and if an arbitrary choice of entity leads to a certain outcome, then that outcome must be true. For example, if we can derive $\exists xHx$, and if we can derive $Hp \to W$ just as easily as $H\ell \to W$, $Hc \to W$, $Hf \to W$, etc., then we can derive W (for example if someone is happy, and the world is a better place if an arbitrarily chosen person is happy, then the world must be a better place).

Every occurrence of the arbitrary individual constant must be existentially instantiated in the antecedent of the implication, because the existential quantifier is being eliminated, so remaining occurrences of the variable would be free which would not result in a WFF. For example, $\forall x(Px \wedge Qx)$ must be instantiated as $(Pa \wedge Qa) \to R$ (or $(Pb \wedge Qb) \to R$, etc.), not as $(Pa \wedge Qx) \to R$ or $(Px \wedge Qa) \to R$.

E∃/EI is typically used after having used rules like E∀/UI, I¬/RAA, I→/CP, or EFSQ to derive the required implication, since these rules allow for the creation of new sentences containing an arbitrary constant. Note that I→/CP is particularly designed for deriving an implication, so it is usually a good choice for trying to derive the implication required for E∃/EI.