solutions

Problem 1: Translation.

a. $\exists x\big(Cx \wedge \exists y(Lxy \wedge Hy)\big)$ or $\exists x \exists y\big(Cx \wedge (Lxy \wedge Hy)\big)$

b. $\forall x(\neg Hx \rightarrow Cx)$

c. $\neg\exists x\big((Cx \wedge Lxh) \wedge Kxh\big)$ or $\forall x \neg\big((Cx \wedge Lxh) \wedge Kxh\big)$ or $\forall x\big((Cx \wedge Lxh) \rightarrow \neg Kxh\big)$

d. $\exists x(Ktx \wedge Hx) \rightarrow \forall y(Lty \rightarrow Cy)$ or $\exists x \forall y\big((Ktx \wedge Hx) \rightarrow (Lty \rightarrow Cy)\big)$

e. $\big((Cd \wedge Cb) \wedge Cs\big) \wedge \forall x\Bigg(\big((x = d \vee x = b) \vee x = s\big) \rightarrow \forall y\Bigg(\Big(\big((y = d \vee y = b) \vee y = s\big) \wedge x \neq y\Big) \rightarrow Kxy\Bigg)\Bigg)$

or without quantifiers $\big((Cd \wedge Cb) \wedge Cs\big) \wedge \Big(\big((Kdb \wedge Kbd) \wedge (Kds \wedge Ksd)\big) \wedge (Kbs \wedge Ksb)\Big)$

f. $\forall x(Kax \rightarrow Cx)$

g. $\exists x \exists y\Big((Lgx \wedge Lgy) \wedge \big(x \neq y \wedge (Cx \wedge Cy)\big)\Big)$

h. $\exists x(Krx \wedge Lrx)$

i. $\neg\exists x(Krx \wedge \neg Lrx)$ or $\forall x \neg(Krx \wedge \neg Lrx)$ or $\forall x(Krx \rightarrow Lrx)$

j. $\exists x\big(Hx \wedge \forall y(Cy \rightarrow Kyx)\big)$ or $\exists x \forall y\big(Hx \wedge (Cy \rightarrow Kyx)\big)$

Problem 2: Valuation in a Model.

a. Various solutions are possible. Sentence 1e being true requires that D'Anna, Boomer, and Six all be Cylons. In addition, sentence 1b being true requires that any entity not in $I(H)$ must be in $I(C)$, while sentences 1a and 1j being true require that $I(H)$ contain at least one entity. (Any entities we put in $I(H)$ could be put into $I(C)$ instead, as long at least one remains in $I(H)$, and further, any entities could be in both sets simultaneously, since there is no restriction that there are no human Cylons!)

$I(C) = \{\text{BOOMER}, \text{D'ANNA}, \text{SIX}\}$ $\qquad$ $I(H) = \{\text{APOLLO}, \text{GAIUS}, \text{HELO}, \text{ROSALYN}, \text{TIGH}\}$

Sentence 1a being true means there must be a Cylon-human pair in $I(L)$. We'll pick $\langle\text{BOOMER}, \text{HELO}\rangle$, to tie into 1c. Further, because sentence 1g is true, Gaius must love two of the Cylons. We'll pick D'Anna and Six, so $\langle\text{GAIUS}, \text{D'ANNA}\rangle$ and $\langle\text{GAIUS}, \text{SIX}\rangle$ must be in $I(L)$:

$$I(L) = \{\langle\text{BOOMER}, \text{HELO}\rangle, \langle\text{GAIUS}, \text{D'ANNA}\rangle, \langle\text{GAIUS}, \text{SIX}\rangle\}$$

To make sure that 1c is false, some Cylon that loves Helo (so far, just Boomer), has to have tried to kill him. This means $\langle\text{BOOMER}, \text{HELO}\rangle \in I(K)$. For sentence 1e to be true, $I(K)$ needs to contain all six pairs combining D'ANNA, BOOMER, and SIX. Further, sentence 1j being true requires that one of the humans be the object of murder attempts by every Cylon. We'll pick Gaius to be that human. In addition, sentence 1f being false requires that there be some non-Cylon that Apollo tried to kill. Since Gaius is a popular choice, we'll add $\langle\text{APOLLO}, \text{GAIUS}\rangle$ to $I(K)$. Sentence 1i must be false as well, which occurs when there is someone Rosalyn tried to kill that she doesn't love. So, we'll have her kill Gaius, too, and make sure she doesn't love him. And in fact, as long as she loves no one at all, sentence 1h will be false, as required:

$$I(K) = \left\{\begin{array}{l} \langle\text{D'ANNA}, \text{BOOMER}\rangle, \langle\text{D'ANNA}, \text{SIX}\rangle, \langle\text{BOOMER}, \text{SIX}\rangle, \\ \langle\text{BOOMER}, \text{D'ANNA}\rangle, \langle\text{SIX}, \text{D'ANNA}\rangle, \langle\text{SIX}, \text{BOOMER}\rangle, \\ \langle\text{D'ANNA}, \text{GAIUS}\rangle, \langle\text{BOOMER}, \text{GAIUS}\rangle, \langle\text{SIX}, \text{GAIUS}\rangle, \\ \langle\text{APOLLO}, \text{GAIUS}\rangle, \langle\text{ROSALYN}, \text{GAIUS}\rangle, \langle\text{BOOMER}, \text{HELO}\rangle, \ldots \end{array}\right\}$$

Finally, to make sure sentence 1d true, we could make sure Tigh never tried to kill a human or doesn't love anyone. Either way is fine, so we'll leave the model as is, since it makes both conditions true.

b. $V_{\mathbb{M}}\Big(\exists x\big(Cx \wedge \exists y(Lxy \wedge Hy)\big)\Big) = 1$ iff $V_{\mathbb{M}}\big(C\alpha \wedge \exists y(L\alpha y \wedge Hy)\big) = 1$ for some individual constant α. Let's consider $\alpha = b$.

$V_{\mathbb{M}}\big(Cb \wedge \exists y(Lby \wedge Hy)\big) = 1$ iff $V_{\mathbb{M}}(Cb) = 1$ and $V_{\mathbb{M}}\big(\exists y(Lby \wedge Hy)\big) = 1$.

$V_{\mathbb{M}}(Cb) = 1$ iff $I(b) \in I(C)$, which is the case iff $\text{B}\textsc{oomer} \in I(C)$. This is the case, so $V_{\mathbb{M}}(Cb) = 1$, so we now only need to show that $V_{\mathbb{M}}\big(\exists y(Lby \wedge Hy)\big) = 1$.

$V_{\mathbb{M}}\big(\exists y(Lby \wedge Hy)\big) = 1$ iff $V_{\mathbb{M}}(Lb\beta \wedge H\beta) = 1$ for some individual constant β. Let's consider $\beta = h$.

$V_{\mathbb{M}}(Lbh \wedge Hh) = 1$ iff $V_{\mathbb{M}}(Lbh) = 1$ and $V_{\mathbb{M}}(Hh) = 1$.

$V_{\mathbb{M}}(Lbh) = 1$ iff $\langle I(b), I(h)\rangle \in I(L)$, which is the case iff $\langle \text{B}\textsc{oomer}, \text{H}\textsc{elo}\rangle \in I(L)$. This is the case, so $V_{\mathbb{M}}(Lbh) = 1$, so we now only need to show that $V_{\mathbb{M}}(Hh) = 1$.

$V_{\mathbb{M}}(Hh) = 1$ iff $I(h) \in I(H)$, which is the case iff $\text{H}\textsc{elo} \in I(H)$. This is the case, so $V_{\mathbb{M}}(Hh) = 1$. This means we have shown everything we need to show, so $V_{\mathbb{M}}\Big(\exists x\big(Cx \wedge \exists y(Lxy \wedge Hy)\big)\Big) = 1$.

c. $V_{\mathbb{M}}\Big(\neg\exists x\big((Cx \wedge Lxh) \wedge Kxh\big)\Big) = 0$ iff $V_{\mathbb{M}}\Big(\exists x\big((Cx \wedge Lxh) \wedge Kxh\big)\Big) = 1$, which is true iff $V_{\mathbb{M}}\big((Cx \wedge Lxh) \wedge Kxh\big) = 1$ for some individual constant α. Let's consider $\alpha = b$.

$V_{\mathbb{M}}\big((Cb \wedge Lbh) \wedge Kbh\big) = 1$ iff $V_{\mathbb{M}}(Cb \wedge Lbh) = 1$ and $V_{\mathbb{M}}(Kbh) = 1$. The latter is true iff $\langle I(b), I(h)\rangle \in I(K)$, which is the case iff $\langle \text{B}\textsc{oomer}, \text{H}\textsc{elo}\rangle \in I(K)$. This is the case, so $V_{\mathbb{M}}(Kbh) = 1$, and we now only need to show that $V_{\mathbb{M}}(Cb \wedge Lbh) = 1$.

$V_{\mathbb{M}}(Cb \wedge Lbh) = 1$ iff $V_{\mathbb{M}}(Cb) = 1$ and $V_{\mathbb{M}}(Lbh) = 1$, which is the case iff $I(b) \in I(C) = 1$ and $\langle I(b), I(h)\rangle \in I(L)$, which is the case iff $\text{B}\textsc{oomer} \in I(C)$ and $\langle \text{B}\textsc{oomer}, \text{H}\textsc{elo}\rangle \in I(L)$. Both of these are true, so $V_{\mathbb{M}}(Cb \wedge Lbh) = 1$. This means we have shown everything we need to show, so $V_{\mathbb{M}}\Big(\neg\exists x\big((Cx \wedge Lxh) \wedge Kxh\big)\Big) = 0$.

Problem 3: Proof. Let $\forall x(Px \leftrightarrow Qx)$ be a WFF of a formal language $\mathscr{L}$, and let $\mathbb{M}$ be a model of $\mathscr{L}$ with a domain of entities $\mathscr{D}$ and an interpretation function I.

Let $I(P) = I(Q)$. The equality of $I(P)$ and $I(Q)$ means they contain exactly the same entities. That is, for any entity ε, $\varepsilon \in I(P)$ iff $\varepsilon \in I(Q)$.

$V_{\mathbb{M}}\big(\forall x(Px \leftrightarrow Qx)\big) = 1$ iff $V_{\mathbb{M}}(P\alpha \leftrightarrow Q\alpha) = 1$ for all individual constants α. Let's choose a completely arbitrary individual constant with no special properties, called $\overline{\alpha}$, and let $I(\overline{\alpha}) = \overline{\varepsilon}$ be whatever entity $\overline{\alpha}$ happens to refer to. For a given entity like $\overline{\varepsilon}$, it must either be in $I(P)$ or not in $I(P)$, so we have two cases to consider:

Case #1: Let $\overline{\varepsilon} \in I(P)$. Since $I(P) = I(Q)$, we also know that $\overline{\varepsilon} \in I(Q)$. Then $I(\overline{\alpha}) \in I(P)$ and $I(\overline{\alpha}) \in I(Q)$, which means $V_{\mathbb{M}}(P\overline{\alpha}) = 1$ and $V_{\mathbb{M}}(Q\overline{\alpha}) = 1$. So, in this case, $V_{\mathbb{M}}(P\overline{\alpha}) = V_{\mathbb{M}}(Q\overline{\alpha})$.

Case #2: Let $\overline{\varepsilon} \notin I(P)$. Since $I(P) = I(Q)$, we also know that $\overline{\varepsilon} \notin I(Q)$. Then $I(\overline{\alpha}) \notin I(P)$ and $I(\overline{\alpha}) \notin I(Q)$, which means $V_{\mathbb{M}}(P\overline{\alpha}) = 0$ and $V_{\mathbb{M}}(Q\overline{\alpha}) = 0$. So, in this case, $V_{\mathbb{M}}(P\overline{\alpha}) = V_{\mathbb{M}}(Q\overline{\alpha})$.

There were only two cases, and one of them must be true, and we derived that $V_{\mathbb{M}}(P\overline{\alpha}) = V_{\mathbb{M}}(Q\overline{\alpha})$ in both, so we must conclude that indeed, $V_{\mathbb{M}}(P\overline{\alpha}) = V_{\mathbb{M}}(Q\overline{\alpha})$, which means that $V_{\mathbb{M}}(P\overline{\alpha} \leftrightarrow Q\overline{\alpha}) = 1$. Our choice of $\overline{\alpha}$ was arbitrary, and nothing in our proof depended on any specific properties of $\overline{\alpha}$ (other than it must be interpretable in $\mathbb{M}$, which we assume is true for all individual constants anyway), so this means that $V_{\mathbb{M}}(P\alpha \leftrightarrow Q\alpha) = 1$ for all possible choices of α, which means that $V_{\mathbb{M}}\big(\forall x(Px \leftrightarrow Qx)\big) = 1$.