LING 230 **Homework #4: Natural Deduction** **Fall 2009**

solutions

Problem 1: Short logical arguments.

a.

1	$X \leftrightarrow Y$	assumption
2	$X \rightarrow Y$	E$\leftrightarrow$, 1
3	$Y \rightarrow X$	E$\leftrightarrow$, 1
4	$X \rightarrow X$	HS, 2, 3
5	$X \leftrightarrow X$	I$\leftrightarrow$, 4, 4

b.

1	$P \wedge Q$	assumption
2	Q	E$\wedge$, 1
3	$Q \vee P$	I$\vee$, 2

c.

1	$J \rightarrow K$	assumption
2	$\neg K$	assumption
3	$\neg J$	MT, 1, 2
4	$\neg J \wedge \neg K$	I$\wedge$, 3, 2

d.

1	$(Qbc \wedge Qca) \wedge \neg Qbc$	assumption
2	$\neg Qbc$	E$\wedge$, 1
3	$Qbc \wedge Qca$	E$\wedge$, 1
4	Qbc	E$\wedge$, 3
5	$\bot$	E$\neg$, 2, 4
6	$\neg Qac$	EFSQ, 5

e.

1	$\neg\exists x Aax$	assumption
2	$\neg\forall y Eye \vee \exists x Aax$	assumption
3	$\neg\forall y Eye$	DS, 2, 1

Problem 2: Longer logical argument.

1	$\neg A \wedge B$	assumption
2	$C \rightarrow \neg B$	assumption
3	$\neg C \leftrightarrow D$	assumption
4	$(E \vee B) \rightarrow (\neg D \vee A)$	assumption
5	B	E$\wedge$, 1
6	$E \vee B$	I$\vee$, 5
7	$\neg D \vee A$	E$\rightarrow$/MP, 4, 6
8	$\neg A$	E$\wedge$, 1
9	$\neg D$	DS, 7, 8
10	$\neg C \rightarrow D$	E$\leftrightarrow$, 3
11	$\neg\neg C$	MT, 10, 9
12	C	$\neg\neg$, 11
13	$\neg B$	E$\rightarrow$/MP, 2, 12
14	$A \vee \neg B$	I$\vee$, 13

Problem 3: Proving semantic validity.

a. Let $\mathbb{M}$ be an arbitrary model in which $V_{\mathbb{M}}(\phi \vee \psi) = 1$ and $V_{\mathbb{M}}(\neg\psi) = 1$. Since $V_{\mathbb{M}}(\phi \vee \psi) = 1$, either $V_{\mathbb{M}}(\phi) = 1$ or $V_{\mathbb{M}}(\psi) = 1$. Call this disjunctive set of conditions $\boxed{\text{A}}$. Since $V_{\mathbb{M}}(\neg\psi) = 1$, we know that $V_{\mathbb{M}}(\psi) = 0$. This contradicts the second condition in $\boxed{\text{A}}$, so the first condition must be the case. Thus, $V_{\mathbb{M}}(\phi) = 1$, and therefore, $\phi \vee \psi, \neg\psi \models \phi$ in $\mathbb{M}$. But since $\mathbb{M}$ is an arbitrary model, we have shown that $\phi \vee \psi, \neg\psi \models \phi$ for any model, which means that DS is semantically valid.

b. Let $\mathbb{M}$ be an arbitrary model in which $V_{\mathbb{M}}(\phi \rightarrow \psi) = 1$ and $V_{\mathbb{M}}(\neg\psi) = 1$. Since $V_{\mathbb{M}}(\phi \rightarrow \psi) = 1$, either $V_{\mathbb{M}}(\phi) = 0$ or $V_{\mathbb{M}}(\psi) = 1$. Call this disjunctive set of conditions $\boxed{\text{A}}$. Since $V_{\mathbb{M}}(\neg\psi) = 1$, we know that $V_{\mathbb{M}}(\psi) = 0$. This contradicts the second condition in $\boxed{\text{A}}$, so the first condition must be the case. Thus, $V_{\mathbb{M}}(\phi) = 0$, which means $V_{\mathbb{M}}(\neg\phi) = 1$ in $\mathbb{M}$. But since $\mathbb{M}$ is an arbitrary model, we have shown that $\phi \rightarrow \psi, \neg\psi \models \neg\phi$ for any model, which means that MT is semantically valid.

c. Let $\mathbb{M}$ be an arbitrary model in which $V_{\mathbb{M}}(\phi \vee \psi) = 1$, $V_{\mathbb{M}}(\phi \rightarrow \chi) = 1$, and $V_{\mathbb{M}}(\psi \rightarrow \chi) = 1$. Since $V_{\mathbb{M}}(\phi \vee \psi) = 1$, either $V_{\mathbb{M}}(\phi) = 1$ or $V_{\mathbb{M}}(\psi) = 1$. Call this disjunctive set of conditions $\boxed{\text{A}}$. Since $V_{\mathbb{M}}(\phi \rightarrow \chi) = 1$, either $V_{\mathbb{M}}(\phi) = 0$ or $V_{\mathbb{M}}(\chi) = 1$. Call this disjunctive set of conditions $\boxed{\text{B}}$. Since $V_{\mathbb{M}}(\psi \rightarrow \chi) = 1$, either $V_{\mathbb{M}}(\psi) = 0$ or $V_{\mathbb{M}}(\chi) = 1$. Call this disjunctive set of conditions $\boxed{\text{C}}$.

Suppose $V_{\mathbb{M}}(\chi) = 0$. This contradicts the second condition in each of $\boxed{\text{B}}$ and $\boxed{\text{C}}$, which means the first condition in both must be the case. Thus, $V_{\mathbb{M}}(\phi) = 0$ and $V_{\mathbb{M}}(\psi) = 0$. But this contradicts both of the conditions in $\boxed{\text{A}}$, leaving us with no possible outcome. This means our supposition that $V_{\mathbb{M}}(\chi) = 0$ must be wrong, so $V_{\mathbb{M}}(\chi) = 1$ in $\mathbb{M}$. But since $\mathbb{M}$ is an arbitrary model, we have shown that $\phi \vee \psi, \phi \rightarrow \chi, \psi \rightarrow \chi \models \chi$ for any model, which means that E$\vee$ is semantically valid.